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SLOW FLOW PAST PERIODIC ARRAYS OF CYLINDERS WITH APPLICATION TO HEAT TRANSFER

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Abstract—Solutions for the slow flow past a square and a hexagonal array of cylinders are determined using a somewhat non-conventional numerical method. The calculated values of the drag on a cylinder as a function of c, the volume fraction of the cylinders, are shown to be in excellent agreement with the corresponding asymptotic expressions for $c \ll 1$ and for $c \to c_{max}$, the maximum volume fraction. These solutions are then used to calculate the average temperature difference between the bulk and the cylinders which are heated uniformly under conditions of small Reynolds and Péclet numbers.

1. INTRODUCTION

The study of the flow past an array of circular cylinders continues to attract interest because of the importance of this configuration in the design of many heat and mass transfer equipments. Different numerical techniques for solving the governing equations have appeared in the literature and these have been summarized by Launder & Massey (1978). One of the major difficulties in obtaining a numerical solution via finite differences arises from the curved boundaries of the domain in which the governing equations are to be solved. Launder & Massey (1978) developed a numerical scheme which uses a polar grid in the vicinity of the cylinder and a Cartesian mesh in the remainder of the flow domain, and then solved the Navier–Stokes equations for a few specific geometries. Surprisingly though, exact results for the drag on a cylinder in a periodic array as a function of c, the volume fraction of the solids, are still not available in the literature even for the simpler case of creeping flow.

In the first part of the present paper we present a numerical technique which is particularly suitable for solving the creeping flow equations and which we shall employ to calculate the drag on a cylinder as a function of c for the case of the two periodic arrays, the square array and the hexagonal array. For very dilute ($c \ll 1$) and for very concentrated ($c \to c_{max}$) arrays, these numerical results for the drag, which are believed to be new, are shown to be in excellent agreement with the corresponding analytical expressions.

In the second part of the paper we consider the problem of heat transfer to the moving fluid from uniformly heated cylinders held fixed in a square array under conditions of small Reynolds and Péclet numbers. Recently, Acrivos *et al.* (1980) examined the corresponding case of heat transfer for heated spheres held fixed in a random array and found that the Péclet number plays a subtle role in such processes even when its magnitude is very small in that, as a consequence of the large temperature gradients that are set up inside the bed, the excess temperature of the particles depends on the details of the flow even to leading order. Since this result carries over to the present problem, the solutions obtained in the first part will be used to calculate this excess temperature for the periodic square array as a function of c. Again, for the dilute arrays, the numerical results thus arrived at are in excellent agreement with the corresponding analytical expressions for small c.

2. FLOW PAST A SQUARE ARRAY OF CYLINDERS

2.1 *The formulation of the problem*

Consider the steady motion of an incompressible viscous fluid through a periodic square array of cylinders each of radius al, with $2l$ being the center-to-center distance distance between two adjacent cylinders. The mean flow velocity of magnitude U is in the x_1-direction. We render all variables dimensionless using l as the characteristic length scale and U as the characteristic velocity and assume that the Reynolds number of the flow is very small. Consequently, the equations of motion reduce to the creeping flow equations

$$\nabla^2\psi = \omega, \quad [1]$$

$$\nabla^2\omega = 0, \quad [2]$$

where ψ and ω are, respectively, the stream function and the vorticity. In view of the symmetry of the flow, the boundary conditions are (see figure 1)

$$\psi = \omega = 0 \text{ on } BC, \quad [3]$$

$$\frac{\partial\psi}{\partial x_1} = \frac{\partial\omega}{\partial x_1} = 0 \text{ on } CD \text{ and on } EF, \quad [4]$$

$$\omega = 0, \quad \psi = 1 \text{ on } DE. \quad [5]$$

In addition, because of the no-slip condition at the surface of the solids, we have that

$$\psi = \frac{\partial\psi}{\partial r} = 0 \text{ on } FB. \quad [6]$$

Note that we have one boundary condition each for ψ and ω along all the boundary lines except along FB where both boundary conditions are specified in terms of ψ.

When solving such problems numerically it has been customary (see Leal & Acrivos, 1969) to obtain a boundary condition in terms of ω on $r = a$ using a Taylor series expansion for ψ, which in the present case gives

$$\psi(a + \Delta r) = \psi_{r=a} + \Delta r\left(\frac{\partial\psi}{\partial r}\right)_{r=a} + \frac{(\Delta r)^2}{2}\left(\frac{\partial^2\psi}{\partial r^2}\right)_{r=a} + 0(\Delta r)^3. \quad [7]$$

Since ψ and its normal derivative vanish at $r = a$, and since $\omega(a) = (\partial^2\psi/\partial r^2)_{r=a}$, [7] reduces

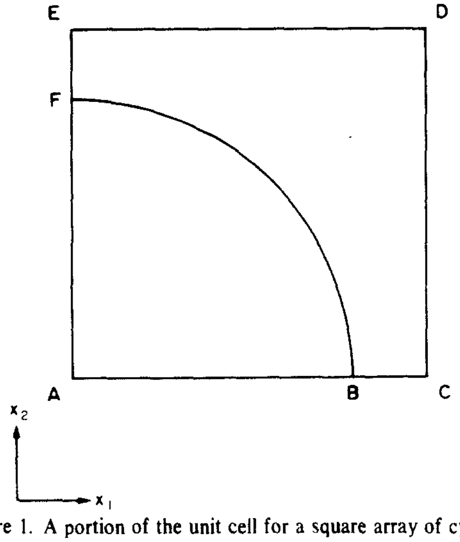

Figure 1. A portion of the unit cell for a square array of cylinders.

to

$$\omega(a) = \frac{2\psi(a+\Delta r)}{(\Delta r)^2} + 0(\Delta r). \qquad [8]$$

The boundary value problem as formulated above can then be solved numerically using suitable iterative methods. However, since the $0(\Delta r)$ term in [8] is proportional to $\partial^3\psi/\partial r^3$ which is of order $1/(1-a)^3$, it is clear that for values of a close to unity, a very fine grid would be required if [8] were used to calculate the vorticity. This difficulty renders the above procedure, rather cumbersome. On the other hand, the method to be described in the next section proved to be very efficient and gave accurate results for the complete range of a.

2.2 *The numerical method*

A general solution of [1] and [2] which satisfies the boundary conditions along BC and EF is given by

$$\omega = 2(4a_1 r + c_1 a^2 r^{-1})\sin\theta + 8\sum_{n=2}^{N}(na_n r^{2n-1} - (n-1)c_n a^{4n-2} r^{1-2n})\sin(2n-1)\theta, \qquad [9]$$

$$\psi = (a_1 r^3 + c_1 a^2 r \ln r + b_1 a^2 r + d_1 a^4 r^{-1})\sin\theta$$
$$+ \sum_{n=2}^{N}\{a_n r^{2n+1} + c_n a^{4n-2} r^{3-2n} + a^2(b_n r^{2n-1} + d_n a^{4n-2} r^{1-2n})\}\sin(2n-1)\theta, \qquad [10]$$

which is exact except for the fact that the infinite series has been truncated to a finite number of terms. The coefficients c_n and d_n in the above equations can be expressed in terms of a_n and b_n using the no-slip boundary conditions [6] at $r = a$. Thus we obtain

$$\omega = 2\left(4a_1 r - \frac{(4a_1 + 2b_1)a^2}{r(2\ln a + 1)}\right)\sin\theta + 8\sum_{n=2}^{N}\sin(2n-1)\theta$$
$$\{na_n r^{2n-1} + (n-1)a^{4n-2} r^{1-2n}(2na_n - (2n-1)b_n)\}, \qquad [11]$$

$$\psi = \left[a_1 r^3\left\{1 - \frac{4\ln r}{2\ln a + 1}\left(\frac{a}{r}\right)^2 + \frac{2\ln a - 1}{2\ln a + 1}\left(\frac{a}{r}\right)^4\right\} + b_1 a^2 r\left\{1 - \frac{2\ln r}{2\ln a + 1} - \right.\right.$$
$$\left.\left.\left(\frac{a}{r}\right)^2 \frac{1}{2\ln a + 1}\right\}\right]\sin\theta + \sum_{n=2}^{N}\left[a_n r^{2n+1}\left\{1 - 2n\left(\frac{a}{r}\right)^{4n-2}\right.\right.$$
$$\left. + (2n-1)\left(\frac{a}{r}\right)^{4n}\right\} + b_n a^2 r^{2n-1}\left\{1 - (2n-1)\left(\frac{a}{r}\right)^{4n-4}\right.$$
$$\left.\left. + 2(n-1)\left(\frac{a}{r}\right)^{4n-2}\right\}\right]\sin(2n-1)\theta. \qquad [12]$$

We are now left with the task of choosing the coefficients a_n and b_n so as to satisfy the remaining boundary conditions in some approximate sense. This is accomplished in the following manner. First we select M number of points $(M > N)$ along the lines CD and DE at which [11] and [12] are to meet the boundary conditions. Since to every point, there are two boundary conditions, we thereby obtain $2M$ linear algebraic equations in $2N$ unknowns. Since $M > N$, these $2M$ equations cannot, in general, be satisfied simultaneously and therefore the unknowns a_n and b_n are determined such that all the $2M$ equations are satisfied in the "least-squares" sense. The method used to accomplish this described by Forsythe *et al.* (1977).

Once the unknown coefficients in the series for ψ have been determined, it is a simple matter to compute the velocity at any point within the unit cell as well as the drag per unit length of a cylinder in the periodic array. The force F per unit length on a cylinder exerted by the fluid moving with mean flow velocity U is given by

$$\frac{F}{\mu U} = a\int_0^{2\pi} (\omega \sin\theta - p\cos\theta)\,d\theta, \qquad [13]$$

where p is the pressure and μ is the viscosity of the fluid. Further, since the solution of the two-dimensional incompressible creeping flow equations can be represented by an analytic function $w(z)$ of the complex variable $z = x_1 + ix_2$ (see Happel & Brenner, 1965) with the real and the imaginary parts of $w(z)$ being, respectively, p/μ and ω, the integral in [13] is related to the imaginary part of the contour integral of w around $|z| = a$, i.e.

$$\frac{F}{\mu U} = \mathrm{Im}\int_{|z|=a} w(z)\,dz. \qquad [14]$$

From the expression for ω (see [11]) we see that $w(z)$ has a simple pole at $z = 0$ and therefore

$$\frac{F}{\mu U} = -\frac{8\pi a^2(2a_1 + b_1)}{(2\ln a + 1)} \qquad [15]$$

2.3 *Results and discussion*

Values of the dimensionless drag $F/\mu U$ for various values of c are listed in table 1 where, in the case of a square array, c is related to a by means of

$$c = \pi a^2/4\,. \qquad [16]$$

Also listed in table 1 are the minimum values of N required to give $F/\mu U$ the accuracy quoted in the table. For most of the calculations, M was chosen in the range $2N$–$5N$ but no significant changes were noted for $M > 2N$. The above results for the drag are also depicted in figure 2 where they are compared to two analytic expressions. The first,

$$\frac{F}{\mu U} = \frac{4\pi}{\ln c^{-1/2} - 0.738 + c - 0.887c^2 + 2.038c^3 + 0(c^4)}, \qquad [17]$$

Table 1

c	F/μU	N	M
0.05	15.56	6	20
0.10	24.83	10	50
0.20	51.53	10	50
0.30	102.90	10	50
0.40	217.89	15	60
0.50	532.55	20	60
0.60	1.763×10^3	25	80
0.70	1.352×10^4	40	90
0.75	1.263×10^5	40	90

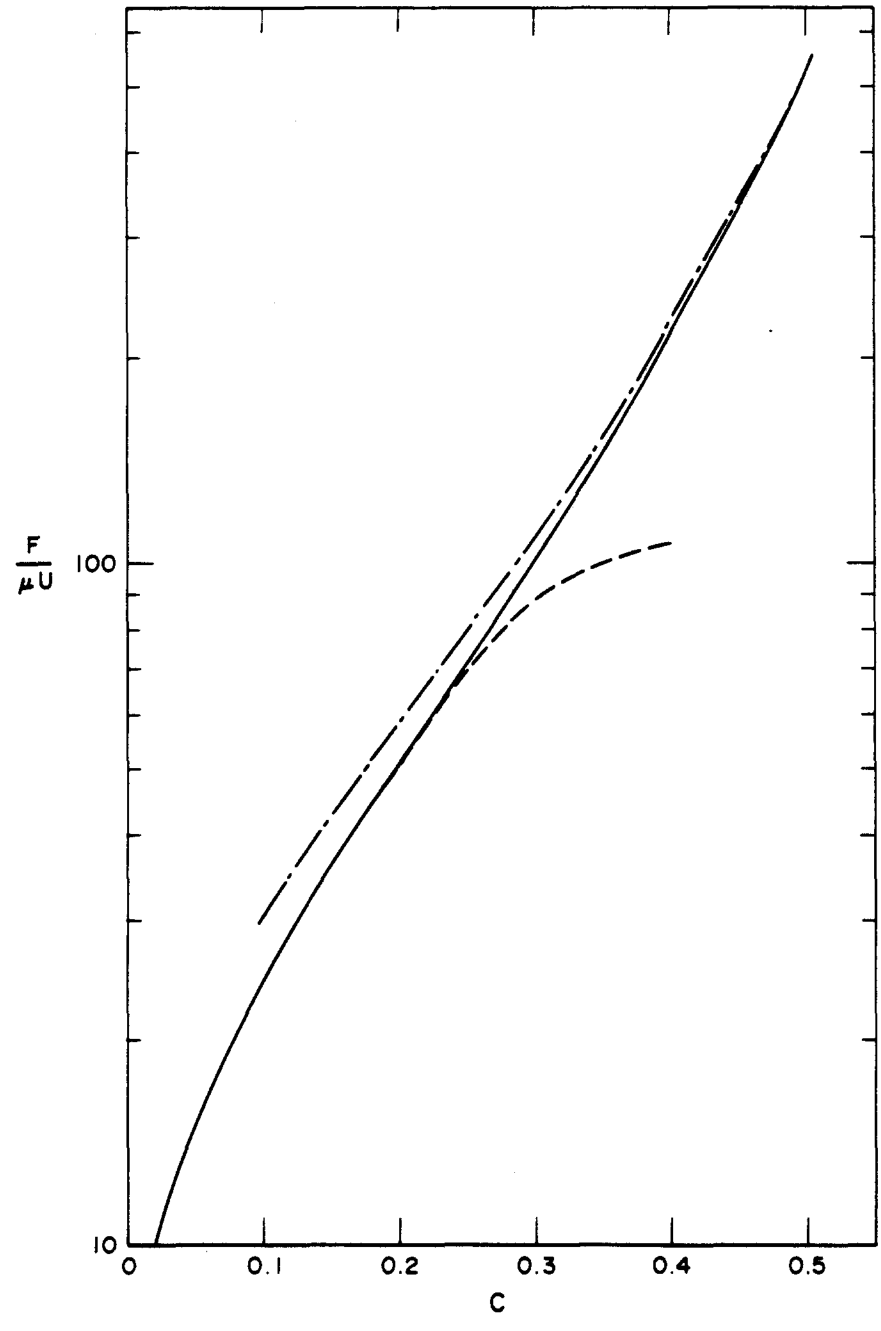

Figure 2. The non-dimensional drag, $F/\mu U$, as a function of the volume fraction, c, for the square array (——, computed values; --- [17], —·—·—, [18]).

which applies to dilute arrays ($c \ll 1$) was recently derived by the present authors (1982a) by extending the earlier analysis of Hasimoto (1959). The second expression is valid for concentrated arrays where most of the pressure drop in the fluid takes place as it flows through a narrow gap between two adjacent cylinders. Application of the usual "lubrication type" approximations gives

$$\frac{F}{\mu U} \sim 9\pi/2\sqrt{(2)}\left\{1-\left(\frac{c}{c_{max}}\right)^{1/2}\right\}^{-5/2}, \quad (c_{max}-c \ll 1) \qquad [18]$$

where c_{max} is the volume fraction of the particles when the cylinders are touching each other

($a = 1$) and equals $\pi/4 = 0.785$ for a square array. As seen in figure 2, the calculated values of the drag are in excellent agreement with, respectively, [17] or [18] when $c \ll 1$ or when $c \to c_{max}$.

It is worth noting that the procedure just described differs from Galerkin's method as used by Snyder & Stewart (1966) to calculate the velocity profiles for the flow through a simple cubic array of spheres because, whereas in the latter the governing differential equations are satisfied in some approximate sense, the trial functions chosen here are exact solutions of these equations.† As a further point of interest we wish to remark that Golub & Gropp (1979) solved Laplace's equation in the domain of figure 1 using a method very similar to ours except that they chose the origin at the corner of the unit cell diagonally across the center of the cylinder (see point D in figure 1) and then determined the unknown coefficients multiplying the trial functions via linear programming rather than by least squares.

The present method can also be used to solve the creeping flow equations in other geometries, e.g. in the hexagonal array to be considered in the next section. Moreover, as will be seen in 4.1.3, the problem of calculating the effective thermal conductivity of a composite material consisting of a regular array of cylinders embedded in a matrix can be treated in a similar manner. In addition to solving these problems for two-dimensional arrays, the above methods can also be extended and applied, in principle, to the corresponding three-dimensional cases.

3. FLOW PAST A HEXAGONAL ARRAY OF CYLINDERS

In this section we present the calculated values of $F/\mu U$ for flow in a hexagonal array, which we obtained by solving [1] and [2], in the domain $\mathscr{D}$ shown in figure 3 with boundary conditions:

$$\omega = \psi = 0 \text{ on } BC \qquad [19]$$

$$\frac{\partial \omega}{\partial x_1} = \frac{\partial \psi}{\partial x_1} = 0 \text{ on } GH,\ CD \qquad [20]$$

$$\psi = \frac{\partial \psi}{\partial n} = 0 \text{ on } BH \qquad [21]$$

$$\omega = 0,\ \psi = \sqrt{3}/2 \text{ on } FG \qquad [22]$$

$$\psi = \sqrt{3}/2,\ \frac{\partial \psi}{\partial n} = 0 \text{ on } DF. \qquad [23]$$

ω and ψ as given by [11] and [12] still satisfy the boundary conditions along GH, HB, and BC, but now the unknown coefficients a_n and b_n are chosen so that the boundary conditions along DF, in addition to those along FG and CD are satisfied in the "least squares" sense.

The non-dimensional drag $F/\mu U$ is still given by [15] and its values for various concentrations c are listed in table 2, where, in the case of a hexagonal array, c is related to a by a means of

$$c = \frac{2}{\sqrt{3}}\pi a^2. \qquad [24]$$

†Sparrow & Loeffler (1959) have treated the case of laminar flow parallel to the axes of cylinders arranged in periodic arrays. These authors obtained a solution to the reduced velocity u^* (see their equation [3]), which satisfies Laplace's equation, by a method very similar to one employed here except that they chose $M = N$ and then satisfied the boundary condition at a set of discrete point equal in number to the number of unknowns in their series solution. We are indebted to the referee for bringing this work to our attention.

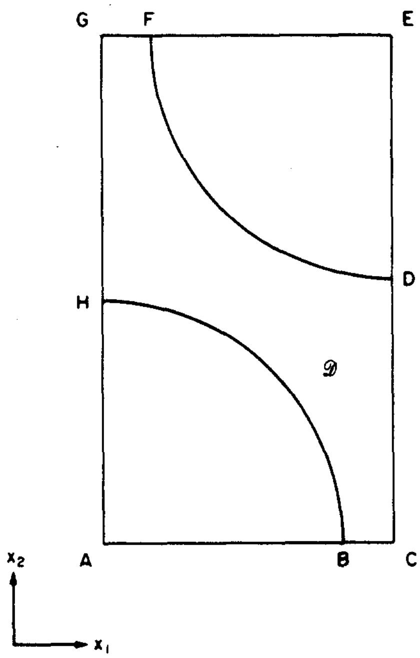

Figure 3. A portion of the unit cell for a hexagonal array of cylinders. The distances AC and AG are, respectively, 1/2 and $\sqrt{3}/2$ units of length.

Table 2

c	F/μU
0.05	15.74
0.10	25.16
0.20	51.15
0.30	96.79
0.40	185.77
0.50	382.2
0.60	901.61
0.70	2.77×10^3
0.80	1.62×10^4
0.85	8.23×10^4

Again, the above results for the drag are depicted in figure 4 where they are compared to two analytic expressions. The first,

$$\frac{F}{\mu U} = \frac{4\pi}{\ln c^{-1/2} - 0.745 + c - \frac{1}{4}c^2 + O(c^4)}, \qquad [25]$$

which applies to dilute arrays ($c \ll 1$) was recently derived by the present authors (1982a), whereas the second expression

$$\frac{F}{\mu U} \sim 27\pi/4\sqrt{2}\left\{1 - \left(\frac{c}{c_{max}}\right)^{1/2}\right\}^{-5/2}, \ (c_{max} - c \ll 1) \qquad [26]$$

applies to concentrated arrays with c_{max} equal to $\pi/2\sqrt{3} = 0.907$. Again there is close correspondence between the analytical and the numerical results.

4. HEAT TRANSFER AT LOW PÉCLET NUMBERS

We now consider the problem of heat transfer to the moving fluid from uniformly heated cylinders held fixed in a periodic array under conditions of small Reynolds and Péclet numbers.

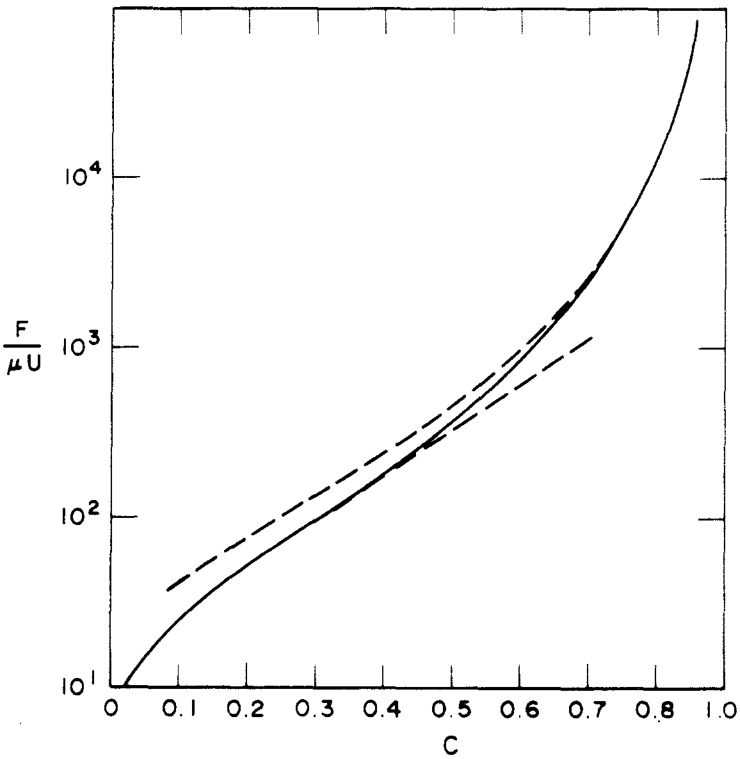

Figure 4. The non-dimensional drag as a function of the volume fraction for the hexagonal array (——, computed values; ---, asymptotic results, [25] and [26])

As mentioned in the introduction, Acrivos *et al.* (1980) have recently examined the corresponding case of heat transfer from heated spheres held fixed in a random array. These authors found that when the Péclet number ϵ is suitably small ($\epsilon^2 \ll c$), the particles are, on average, hotter than the bulk by an amount given by

$$\Delta T = \frac{qa^{*2}}{k_f}\left[\frac{5\alpha+1}{15\alpha} + \frac{\sqrt{2}}{3}c^{1/2} + A(\alpha)c + 0(c^{3/2}\ln c)\right] \; (c \ll 1) \qquad [27]$$

where q is the volumetric heating rate of spheres, a^* is the radius of each sphere, k_f is the thermal conductivity of the fluid, and α is the ratio of thermal conductivities k_s/k_f. The constant A as a function of α is given in figure 1 of their paper. Our goal here is to give the corresponding results for ΔT for the case of two periodic arrays of cylinders, the square array and the hexagonal arrays.

4.1 *The square array*

4.1.1 *The formulation of the problem.* Consider again the periodic square array of Section 2. Since, in general, the average temperature difference ΔT between the solids and the bulk will depend on the direction of the mean flow, we restrict our discussion to the case where the mean flow is in the x_1-direction with magnitude U. We shall also assume that each cylinder is uniformly heated with volumetric rate q. Non-dimensionalizing the temperature with qa^2l^2/k_f, distances with l, and velocities with U and neglecting the viscous heat dissipation in the steady energy balance we obtain

$$\nabla^2 T = \begin{cases} \dfrac{\epsilon}{a} u_i \dfrac{\partial T}{\partial x_i} & \text{within the fluid,} \\ -1/a^2\alpha & \text{inside the particles,} \end{cases} \qquad [28]$$

where ϵ is the Péclet number defined by

$$\epsilon = \frac{alU\rho c_p}{k_f}. \qquad [29]$$

Due to the heat sources inside the particles, a mean temperature gradient, parallel to the direction of the mean flow, will be established in the bed and the magnitude of this gradient, as obtained from an overall energy balance, is

$$\left\langle \frac{\partial T}{\partial x_1} \right\rangle = \frac{\pi}{4}\frac{a}{\epsilon}. \qquad [30]$$

Clearly, for a given value of a, the non-dimensional temperature gradient in the bed is $0(\epsilon^{-1})$ as $\epsilon \to 0$. Following Acrivos *et al.* (1980) we therefore expand the temperature in power series of ϵ, i.e. we let

$$T(\mathbf{x};\epsilon) = \frac{1}{\epsilon}T_{-1} + T_0 + \epsilon T_1 + \ldots, \qquad [31]$$

which when substituted in the energy equation [28] leads to the sequence of problems:

$$\epsilon^{-1}: \quad \nabla^2 T_{-1} = 0 \quad \text{(everywhere)}. \qquad [32]$$

$$\epsilon^{0}: \quad \nabla^2 T_0 = \begin{cases} \dfrac{1}{a} u_j \dfrac{\partial T_{-1}}{\partial x_j} & \text{within the fluid.} \\ -1/a^2\alpha & \text{inside the particles.} \end{cases} \qquad [33]$$

$$\epsilon^{1}: \quad \nabla^2 T_1 = \begin{cases} \dfrac{1}{a} u_j \dfrac{\partial T_0}{\partial x_j} & \text{within the fluid,} \\ 0 & \text{inside the particles.} \end{cases} \qquad [34]$$

From the overall energy balance [30] we see that the mean temperature gradient in the x_1-direction arises only from T_{-1} and that the magnitude of this gradient is $\pi a/4$. We also note that T_{-1} and T_1 are anti-symmetric in x_1 and that T_0 a symmetric function in x_1 and in $(1-x_1)$ (see figure 1). Further, since ∇T is periodic, it can be seen that T_{-1} and T_1 must be constant along $x_1 = 0$ and $x_1 = 1$. Thus if we set the temperature of the origin equal to zero, then we have following boundary conditions:

$$T_{-1} = T_1 = 0, \quad \frac{\partial T_0}{\partial x_1} = 0 \text{ on } x_1 = 0 \qquad [35]$$

$$T_{-1} = \pi a/4, \quad T_1 = \frac{\partial T_0}{\partial x_1} = 0 \text{ on } x_1 = 1, \qquad [36]$$

plus the insulating conditions

$$\frac{\partial T_{-1}}{\partial x_2} = \frac{\partial T_0}{\partial x_2} = \frac{\partial T_1}{\partial x_2} = 0 \text{ on } x_2 = 0, x_2 = 1. \qquad [37]$$

In addition, the temperature and the heat flux must be continuous at $r = a$, i.e. for each term in

[31], we require that

$$T_{r=a+} = T_{r=a-} \qquad [38]$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=a+} = \alpha\left(\frac{\partial T}{\partial r}\right)_{r=a-} . \qquad [39]$$

Finally, we define the non-dimensional excess temperature of the solid as

$$\Delta T = \langle T \rangle_{\text{solid}} - \langle T \rangle_{\text{unit cell}} , \qquad [40]$$

where the bracket $\langle\ \rangle$ denotes the averaging operator.

Since T_{-1} and T_1 are anti-symmetric about $x_1 = 0$, they do not contribute to ΔT, i.e.

$$\Delta T = \Delta T_0 + 0(\epsilon^2) . \qquad [41]$$

Thus in order to determine ΔT to the leading order, we must solve the corresponding problems for T_{-1} and T_0. We wish to point out here that T_{-1} is the solution of the well known problem of heat conduction in a periodic array of cylinders which we shall treat in a later section. First, though, we consider the simpler case of equal thermal conductivities.

4.1.2 *The case of equal conductivities* ($\alpha = 1$). In this case, T_{-1} is a linear function of x_1 given by

$$T_{-1} = \frac{\pi}{4} a x_1 , \qquad [42]$$

and [33] for T_0 reduces to

$$\nabla^2 T_0 = \begin{cases} \dfrac{\pi}{4} u_1 & \text{within the fluid} \\ -1/a^2 & \text{inside the particles} \end{cases} \qquad [43]$$

with u_1 given by the solution obtained in Section 2. The above equation was solved using a Fourier-series technique and the computed values of ΔT_0 as a function of the volume function c of the particles are given in figure 5 where they are compared with the analytic expression

$$\Delta T_0 = \frac{1}{2}(\ln c^{-1/2} - 0.488) + \frac{3}{8}c - \frac{1}{8}c^2 + \frac{1}{Q}\Big[0.076 - \frac{c}{2}(\ln c^{-1/2} - 0.238) + \frac{c^2}{4}(\ln c^{-1/2} - 2.131) + 0.673c^3 + 0(c^4)\Big] , \qquad [44]$$

with

$$Q = \ln c^{-1/2} - 0.738 + c - 0.887c^2 + 2.039c^3 + 0(c^4)$$

which was recently derived by the present authors (1982b).

4.1.3 *The case of unequal conductivities.* We now return to the case of arbitrary α. In this case, we must first solve T_{-1}. This problem of determining the solution to Laplace's equation in a periodic array of particles is a classic one starting with the work of Rayleigh who, in 1892, gave an expression for the effective thermal conductivity of a dilute periodic array. Recent

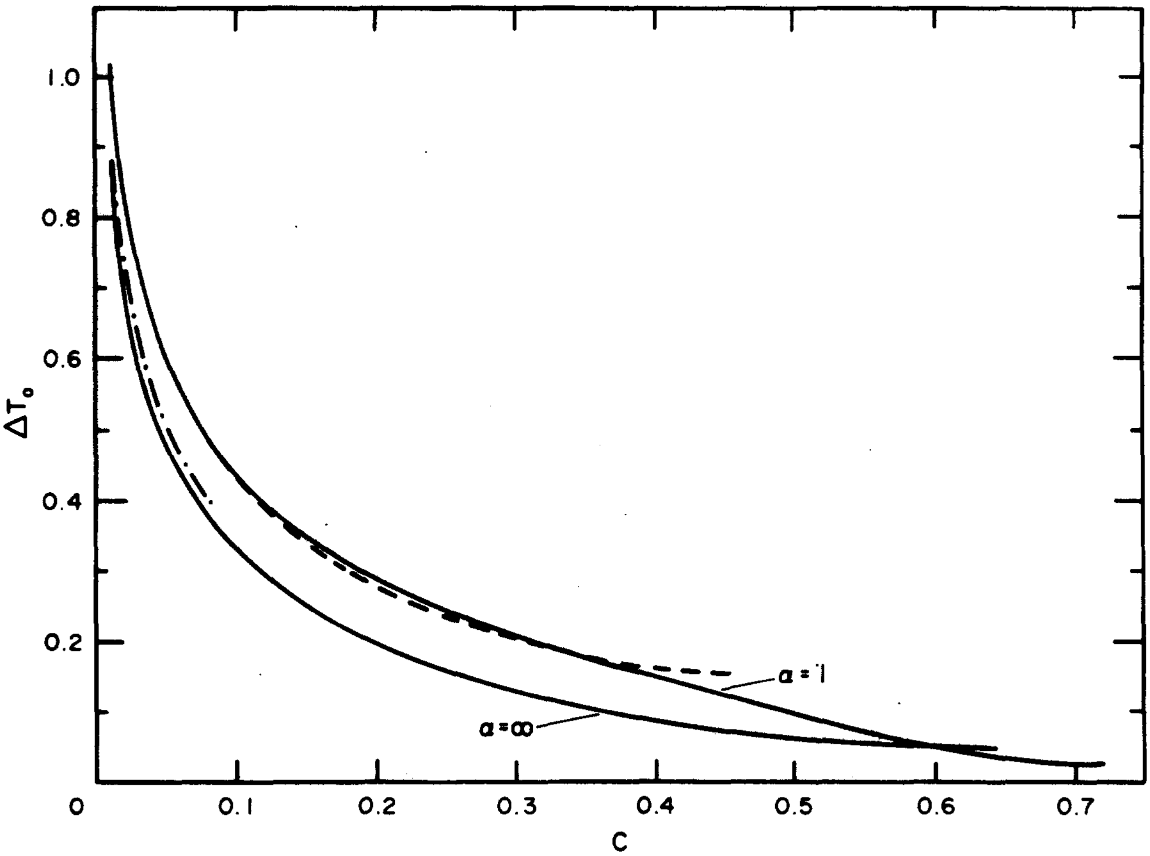

Figure 5. The average temperature difference, ΔT_0, as a function of concentration for the square array (——, computed values; ---, [44]; ----, [47]).

contributions are those of Keller & Sachs (1964) who calculated numerically, using a finite-difference scheme, the values of the effective thermal conductivity for a square array of perfectly conducting cylinders ($\alpha = \infty$) and of Perrins *et al.* (1979) who extended Rayleigh's method and obtained the effective conductivities for the square and the hexagonal arrays. Unfortunately, since these earlier investigators only reported the effective conductivities it became necessary to resolve this problem because the full temperature profiles are needed for our purposes. Following the same procedure as in section 2 we therefore express the general solution, truncated to a series with only a finite number of terms, which satisfies the boundary conditions on $x_1 = 0$, $x_2 = 0$, and on $r = a$ in figure 1, as

$$T_{-1} = \begin{cases} \sum_{n=1}^{N} a_n \left(r^{2n-1} + \frac{1-\alpha}{1+\alpha} a^{4n-2} r^{1-2n} \right) \cos(2n-1)\theta, & r > a, \\ \sum_{n=1}^{N} \frac{2a_n}{1+\alpha} r^{2n-1} \cos(2n-1)\theta, & r \leq a, \end{cases} \qquad [45]$$

and determine the coefficients a_n such that the remaining boundary conditions are satisfied approximately in the sense of the lease squares. The expression for the effective thermal conductivity then becomes

$$\frac{k_{ef}}{k} = \frac{4}{\pi a} \sum_{n=1}^{N} (-1)^n a_n \left(1 - \frac{1-\alpha}{1+\alpha} a^{4n-2} \right), \qquad [46]$$

but since the computed values of the effective conductivities were found to be essentially identical to those reported in the literature, these will not be presented here.

Once the full temperature profiles for T_{-1} are determined, [33] can be solved for any arbitrary α. However, we shall only present the results for the specific case of perfectly conducting cylinders ($\alpha = \infty$). Here, the temperature inside a particle is uniform and accordingly the temperature at $r = a$ can be set equal to zero. Again, the method of Section 2 could have been used to obtain an expression for T_0 which, in this case, would have involved the product of two trigonometric series but, unfortunately, the subsequent evaluation of $\langle T_0 \rangle$ via the numerical integration of T_0 over the unit cell would have been computationally inefficient. Therefore, a conventional finite difference method was employed and specifically the usual five-point difference formula for Poisson's equation in combination with the Successive Over Relaxation (SOR) method. The boundary conditions were also satisfied to $0(h^2)$, $-$ h being the grid size.

The results for the computed values of ΔT_0 are also given in figure 5 where they are shown to be in excellent agreement with the analytical expression

$$\Delta T_0 = \frac{1}{2}\left\{\ln c^{-1/2} - 0.738 + \frac{1}{4\alpha}\right\} + \frac{0.076}{\ln c^{-1/2} - 0.738} + 0(c)\,, \qquad [47]$$

which was recently derived by the present authors (1982b).

4.2 *The hexagonal array*

The equations derived in section 4.1 also apply to the hexagonal array with a few minor modifications. Thus in lieu of [30] and [36] we have now the following equations

$$\left\langle \frac{\partial T}{\partial x_1} \right\rangle = \frac{2\pi}{\sqrt{3}}\frac{a}{\epsilon}\,, \qquad [48]$$

$$T_{-1} = \frac{\pi a}{\sqrt{3}}, \quad T_1 = 0 \text{ on } x_1 = \frac{1}{2} \qquad [49]$$

and now the governing differential equations must be solved in the domain shown in figure 3. The computed values for ΔT_0 as a function of c for two special cases $\alpha = 1$ and $\alpha = \infty$ are given in figure 6 where, once again, these results are compared with the analytical expressions

$$\Delta T_0 = \frac{1}{2}\left(\ln c^{-1/2} - 0.745 + \frac{1}{4\alpha}\right) + \frac{0.073}{\ln c^{-1/2} - 0.745} + 0(c), \text{ any } \alpha \qquad [50]$$

and

$$\Delta T_0 = \frac{1}{2}(\ln c^{-1/2} - 0.495) + \frac{3}{8}c - \frac{1}{8}c^2 + \frac{1}{Q}\Big[0.073 - \frac{c}{2}(\ln c^{-1/2} - 0.225)$$
$$+ \frac{c^2}{4}(\ln c^{-1/2} - 1.62) + \frac{7}{32}c^3 + 0(c^4)\Big],\ \alpha = 1 \qquad [51]$$

with

$$Q = \ln c^{-1/2} - 0.745 + c - \frac{1}{4}c^2 + 0(c^4)$$

which were recently derived by the present authors (1982b). These expressions for ΔT_0 for the hexagonal array and [44] and [47] for the square array are valid for $\epsilon^2 \ll c \ll 1$. When $c \ll \epsilon^2 \ll 1$, the nature of solution for T is quite different. As discussed by Acrivos *et al.* (1980), under these

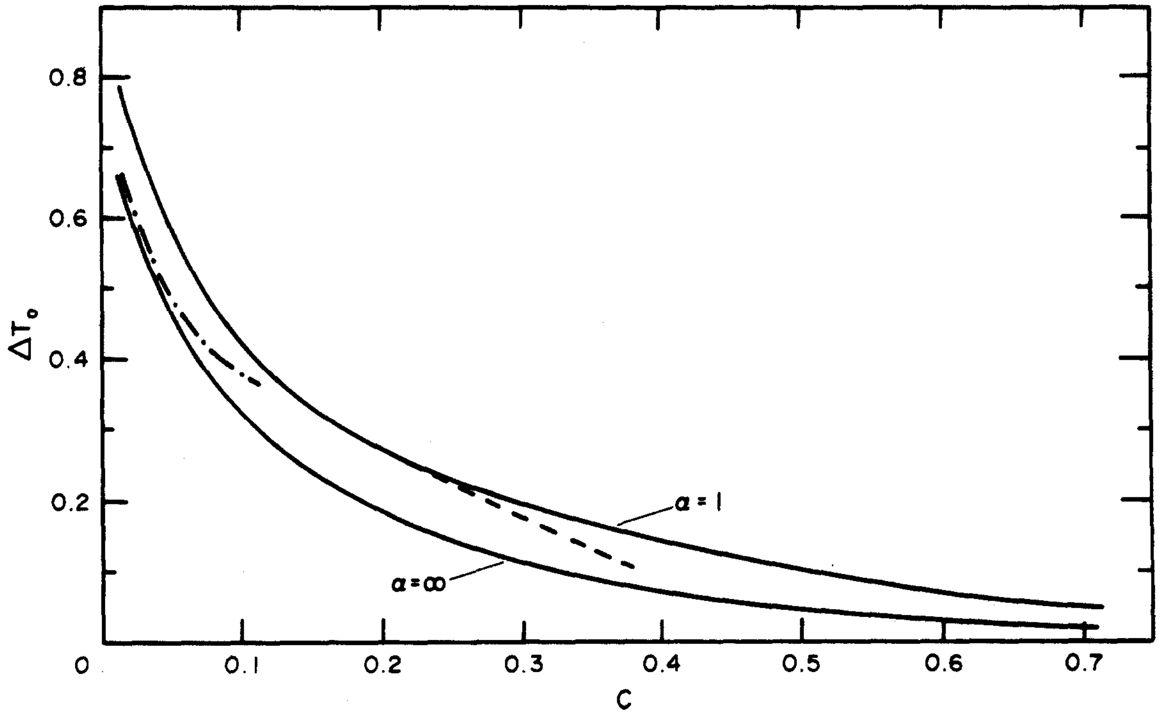

Figure 6. The average temperature difference, ΔT_0, as a function of the volume fraction for the hexagonal array (——, computed values; —·—·—, [50]; ----, [51]).

conditions each particle effectively behaves as an isolated particle and it can be shown very easily that ΔT in this case is $O(\ln \epsilon)$.

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